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Nuclear Physics B 889 (2014) 778–800

NUCLEAR PHYSICS B

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Factorization of radiative leptonic decays of B^- and D^- mesons

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Received 1 September 2014; accepted 30 October 2014

Available online 11 November 2014

Editor: Hong-Jian He

Abstract

In this work, we study the factorization of the radiative leptonic decays of B^- and D^- mesons, the contributions of the order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ are taken into account. The factorization is proved to be valid explicitly at the order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$. The hard kernel is obtained. The numerical results are calculated using the wave-function obtained in relativistic potential model. The $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contribution is found to be very important, the correction to the decay amplitudes of $B^- \to \gamma e \bar{\nu}$ is about 20%–30%. For D mesons, the $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contributions are more important.

1. Introduction

The study of the heavy meson decays is an important field in high energy physics. In recent years, both experimental and theoretical studies have been improved greatly [1–3]. However, the limitation in understanding and controlling the non-perturbative effects in strong interaction is so far still a problem. Varies theoretical methods on how to deal with the non-perturbative effects have been developed. An important approach is to separate the hard and soft physics which is known as factorization [4,5]. This method has been greatly developed in recent years [6].

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http://dx.doi.org/10.1016/j.nuclphysb.2014.10.027

The idea of factorization is to absorb the infrared (IR) behaviour into the wave-function, the matrix element can be written as the convolution of wave-function and hard kernel

$$F = \int dk \Phi(k) T_{\text{hard}}(k) \tag{1}$$

The wave-function should be determined by non-perturbative methods.

The radiative leptonic decay of heavy mesons provides a good opportunity to study the factorization approach, where strong interaction is involved only in one hadronic external state. Except for that, with a photon emitted out, more details about the wave-function of the hadronic bound state can be exploited. Many works has been done on the factorization of this decay mode. In Ref. [7], the 1-loop QCD correction is calculated in the large energy effective theory, and they found the factorization will depend on the transverse momentum. In Refs. [8] and [9], factorization is proved in leading order of $1/m_Q$ expansion in the frame of QCD factorization [4,5], where the heavy quark is treated in the heavy quark effect theory (HQET) [3,10]. In Refs. [11,12], the factorization is constructed using the soft-collinear effective theory (SCET) [13,14].

In this work, factorization in the radiative leptonic decays of heavy mesons is revisited. The work of Refs. [8,9] is extended by taking into account the contributions of the order of $O(\Lambda_{\text{QCD}}/m_Q)$. The factorization is proved to be still valid explicitly. We also find the factorization is valid at any order of $O(\Lambda_{\text{QCD}}/m_Q)$. The numerical results shows that, the $O(\Lambda_{\text{QCD}}/m_Q)$ correction is very important for the B and D mesons, the correction can be as large as 20%–30%.

The remainder of the paper is organized as follows. In Section 2, we discuss the kinematic of the radiative decay and the wave-function. In Section 3, we present the factorization at tree level. In Section 4, the 1-loop corrections of the wave-function are discussed. The factorization at 1-loop order is presented in Section 5. In Section 6, we briefly discuss the resummation of the large logarithms. The numerical results are presented in Section 7. And Section 8 is a summary.

2. The kinematic

The B or D meson is constituted with a quark and an anti-quark, where one of the quarks is a heavy quark, and the other is a light quark. The Feynman diagrams at tree level of the radiative leptonic decay can be shown as Fig. 1. The contribution of Fig. 1d is suppressed by a factor of $1/{M_w}^2$, and can be neglected. The amplitudes of Fig. 1a, b and c can be written as

$$\begin{aligned}
\mathcal{A}_a^{(0)} &= \frac{-ie_q G_F V_{Qq}}{\sqrt{2}} \bar{q}(p_{\bar{q}}) \not{\varepsilon}_\gamma^* \frac{\not{p}_\gamma - \not{p}_q}{2p_\gamma \cdot p_{\bar{q}}} P_L^\mu Q(p_Q)(l P_{L\mu} \bar{\nu}) \\
\mathcal{A}_b^{(0)} &= \frac{-ie_Q G_F V_{Qq}}{\sqrt{2}} \bar{q}(p_{\bar{q}}) P_L^\mu \frac{\not{p}_Q - \not{p}_\gamma + m_Q}{2p_Q \cdot p_\gamma} \not{\varepsilon}_\gamma^* Q(p_Q)(l P_{L\mu} \bar{\nu}) \\
\mathcal{A}_c^{(0)} &= \frac{-eG_F V_{Qq}}{\sqrt{2}} \bar{q}(p_{\bar{q}}) P_L^\mu Q(p_Q) \left(l \not{\varepsilon}_\gamma^* \frac{i(\not{p}_\gamma + \not{p}_l + m_l)}{2(p_\gamma \cdot p_l)} P_{L\mu} \bar{\nu} \right)
\end{aligned} \tag{2}$$

where $p_{\bar{q}}$ and p_Q are the momenta of the anti-quark $\bar{q}$ and quark Q, respectively, p_γ, p_l and p_ν are the momenta of photon, lepton and neutrino, ε_γ denotes the polarization vector of photon, and P_L^ν is defined as $\gamma^\mu(1-\gamma_5)$.

We work in the rest-frame of the meson, and we choose the frame such that the direction of the photon momentum is on the opposite z axis, so the momentum of the photon can be written as $p_\gamma = (E_\gamma, 0, 0, -E_\gamma)$, with $0 \le E_\gamma \le m_Q/2$.

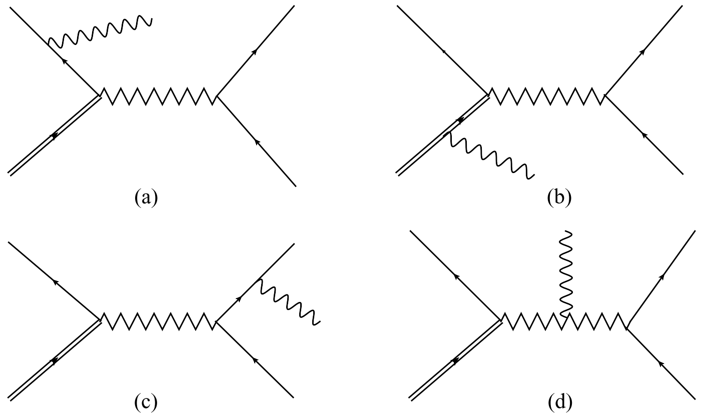

Fig. 1. Tree level amplitudes, the double line represents the heavy quark propagator but not the HQET propagator.

To study the factorization, we consider the state of two free quark and anti-quark at first. The wave-function of the two quark and anti-quark state is defined as

$$\Phi(k_q, k_Q) = \int d^4x d^4y \exp(ik_q \cdot x) \exp(ik_Q \cdot y) \langle 0|\bar{v}_{\bar{q}}(x)[x, y]u_Q(y)|\bar{q}Q\rangle \tag{3}$$

where $[x, y]$ denotes the Wilson line [15]. And the matrix element is defined as

$$F = \langle\gamma|\bar{v}_{\bar{q}}(x)P_L^{\mu}u_Q(y)|\bar{q}Q\rangle \tag{4}$$

The prove of factorization is to prove that, up to 1-loop order, the matrix element can be written as the convolution of the wave-function Φ and a hard-scattering kernel T, where T is IR-finite and independent of the external state.

3. Tree level factorization

We start with the matrix elements at tree level. Using the definition of the wave-function in coordinate space

$$\Phi_{\alpha\beta}(x, y) = \langle 0|\bar{q}_\alpha(x)[x, y]Q_\beta(y)\big|\bar{q}^S(p_{\bar{q}}), Q^s(p_Q)\big\rangle \tag{5}$$

where S and s are spin labels of $\bar{q}$ and Q, respectively. We find

$$\Phi^{(0)}_{\alpha\beta}(k_{\bar{q}}, k_Q) = (2\pi)^4\delta^4(k_{\bar{q}} - p_{\bar{q}})(2\pi)^4\delta^4(k_Q - p_Q)\bar{v}_\alpha(p_{\bar{q}})u_\beta(p_Q) \tag{6}$$

And then the matrix element can be written as

$$F^{(0)} = \int \frac{d^4k_{\bar{q}}}{(2\pi)^4}\frac{d^4k_Q}{(2\pi)^4}\Phi^{(0)}(k_{\bar{q}}, k_Q)T^{(0)}(k_{\bar{q}}, k_Q) = \Phi^{(0)} \otimes T^{(0)} \tag{7}$$

With Eqs. (2) and (7), we obtain the hard scattering kernel at tree level as

$$T_a^{(0)} = -e_q\frac{\not{\varepsilon}^*_\gamma \not{p}_\gamma - 2\varepsilon^*_\gamma \cdot k_q}{2p_\gamma \cdot k_{\bar{q}}}P_L^{\mu}, \qquad T_b^{(0)} = -e_Q P_L^{\mu}\frac{-\not{p}_\gamma\not{\varepsilon}^*_\gamma}{2k_Q \cdot p_\gamma} \tag{8}$$

In the expressions above, we have already assumed to consider the kinematical region $E_\gamma \sim m_Q$. The polarization vector of the photon does not have 0-component, as a result

$(\not{k}_Q + m_Q)\not{\varepsilon} u_Q/2p_\gamma \cdot p_Q$ is an order of $O(\Lambda^2_{\rm QCD}/m_Q^2)$ contribution and is neglected. The remaining terms of $T_b^{(0)}$, and the transverse part of $T_a^{(0)}$, which is $2e_q\varepsilon \cdot k_{\bar q} P_L^\mu/2p_\gamma \cdot k_{\bar q}$ are order of $O(\Lambda_{\rm QCD}/m_Q)$ contributions.

Exchange the Lorentz index in $\mathcal{A}_c$, we obtain

$$\mathcal{A}_c^{(0)} = \frac{-ieG_F V_{Qq}}{\sqrt{2}} \bar q(p_{\bar q}) P_L^\mu \frac{(\not{p}_\gamma \not{\varepsilon}^*_\gamma + 2\varepsilon^*_\gamma \cdot (p_Q + p_{\bar q} - p_\nu))}{2(p_\gamma \cdot (p_Q + p_{\bar q} - p_\nu))} Q(p_Q)(l P_{L\mu}\bar\nu) \tag{9}$$

We find

$$F_c^{(0)} = -e\bar v_{\bar q} P_L^\mu \frac{\not{p}_\gamma \not{\varepsilon}^*_\gamma + 2\varepsilon \cdot (p_Q + p_{\bar q} - p_\nu)}{2p_\gamma \cdot (p_Q + p_{\bar q} - p_\nu)} u_Q$$

$$T_c^{(0)} = -eP_L^\mu \frac{\not{p}_\gamma \not{\varepsilon}^*_\gamma + 2\varepsilon \cdot (k_Q + k_{\bar q} - p_\nu)}{2p_\gamma \cdot (k_Q + k_{\bar q} - p_\nu)} \tag{10}$$

This term is also an order of $O(\Lambda_{\rm QCD}/m_Q)$ contribution.

4. 1-loop correction of wave-function

The expansion of the decay amplitude can be written as [8]

$$F = F^{(0)} + F^{(1)} + \ldots = \Phi^{(0)} \otimes T^{(0)} + \Phi^{(1)} \otimes T^{(0)} + \Phi^{(0)} \otimes T^{(1)} + \ldots \tag{11}$$

At the 1-loop level, the amplitude can be written as

$$F^{(1)} = \Phi^{(1)} \otimes T^{(0)} + \Phi^{(0)} \otimes T^{(1)} \tag{12}$$

The 1-loop corrections of $\Phi \otimes T$ come from the QCD interaction and the Wilson line. The later can be written as [8,15]

$$[x, y] = \exp\left[ig_s \int_y^x d^4z z_\mu A^\mu(z) \right] = \sum_n \frac{(ig_s)^n}{n!} \prod_i^n \int_y^x d^4 z_i z_{i\mu} A^\mu(z_i) \tag{13}$$

The corrections are shown in Fig. 2.

We use $\Phi_q^{(1)}$ to represent the correction with the gluon from the Wilson line connected to the light quark external leg. So the correction in Fig. 2a can be written as

$$\Phi_q^{(1)}(k_{\bar q}, k_Q) = \int d^4x \int d^4y e^{ik_{\bar q}\cdot x} e^{ik_Q \cdot y} \langle 0| \bar q_{\bar q}(x) ig_s \int_y^x dz z_\mu A^\mu(z) Q(y)$$

$$\times ig_s \int d^4x_2 \bar q_{\bar q}(x_2) \not{A}(x_2) q_{\bar q}(x_2) \big| \bar q^S(p_{\bar q}), Q^s(p_Q)\big\rangle \tag{14}$$

After the integration, the result is

$$\Phi_q^{(1)} \otimes T^{(0)}$$

$$= ig_s^2 C_F \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \bar v_{\bar q} \gamma^\rho \frac{(\not{l} + \not{p}_{\bar q} - m_{\bar q})}{(l + p_{\bar q})^2 - m_{\bar q}^2} \int_0^1 d\alpha \left(\frac{\partial T^{(0)}}{\partial k_q^\rho} - \frac{\partial T^{(0)}}{\partial k_Q^\rho} \right)\bigg|_{k_q = k', k_Q = K'} u_Q$$

$$k' = p_{\bar q} + \alpha l, \quad K' = p_Q - \alpha l \tag{15}$$

The procedure of the integration can be found in Appendix A.

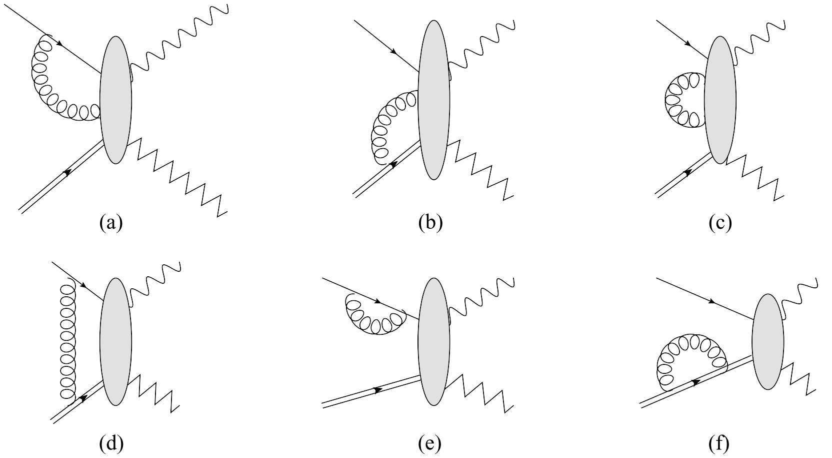

Fig. 2. The 1-loop correction of wave-functions. $\Phi^{(1)} \otimes T_a^{(0)}$ and $\Phi^{(1)} \otimes T_b^{(0)}$ are established in this figure.

Similar to Φ_q, the correction in Fig. 2b can be written as

$$\begin{aligned}
&\Phi_Q^{(1)} \otimes T^{(0)} \\
&= -iC_F g_s^2 \int_0^1 d\alpha \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \bar{v}_{\bar{q}} \left(\frac{\partial T^{(0)}}{\partial k_q^\rho} - \frac{\partial T^{(0)}}{\partial k_Q^\rho} \right) \frac{(\not p_Q - \not l + m_Q)}{(p_Q - l)^2 - m_Q^2} \gamma^\rho u_Q \bigg|_{k_q = k', k_Q = K'} \\
&k' = p_{\bar{q}} + \alpha l, \quad K' = p_Q - \alpha l
\end{aligned} \tag{16}$$

We use Φ_{Wfc} to denote the correction shown in Fig. 2c. We find

$$\Phi_{\mathrm{Wfc}}^{(1)} \otimes T^{(0)} = -\frac{g_s^2 C_F}{2} \int \frac{d^d l}{(2\pi)^d} \int_0^1 d\alpha \int_0^1 d\beta \frac{1}{l^2} \bar{v}_{\bar{q}} \left(\frac{\partial}{\partial k_q} - \frac{\partial}{\partial k_Q} \right)^2 T^{(0)} \bigg|_{k_q = k', k_Q = K'} u_Q \tag{17}$$

The corrections shown in Fig. 2d, 2e and 2f can be denoted as Φ_{box}, Φ_{extQ} and Φ_{extq}. We find that, they have the same forms as the free particle 1-loop QCD corrections.

5. 1-loop factorization

For simplicity, we denote

$$x = m_Q^2, \qquad y = 2p_Q \cdot p_\gamma, \qquad z = 2p_\gamma \cdot p_{\bar{q}}, \qquad w = 2p_Q \cdot p_{\bar{q}} \tag{18}$$

The definitions of x, y and z are the same as Ref. [8] at order $O(\Lambda_{\mathrm{QCD}}/m_Q)^0$, while w is a new scalar that appears at the order of $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contributions which will be shown later, it represents the effect of the transverse momentum.

To calculate the hard-scattering amplitude, we need to calculate all 1-loop corrections of F_a, F_b and F_c. We take a small mass m_q for the light quark to regulate the collinear IR divergences. The soft IR divergences will not appear explicitly in this factorization procedure. We use $\overline{\mathrm{MS}}$ [19]

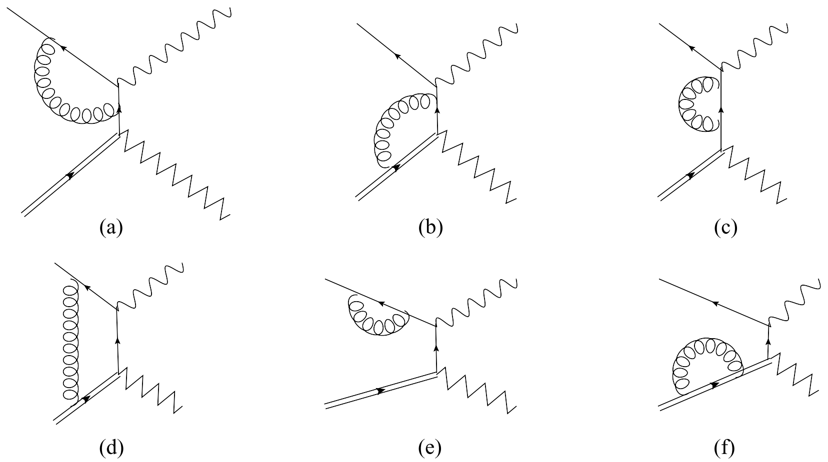

Fig. 3. 1-loop QCD correction of F_a.

scheme to regulate the ultraviolet (UV) divergences, in $D = 4 - \epsilon$ demission, we define N_{UV} as

$$N_{\text{UV}} = \frac{2}{\epsilon} - \gamma_E + \log(4\pi) \tag{19}$$

We take the factorization scale the same as the renormalization scale, so we use the same μ in $F^{(1)}$ and $\Phi^{(1)} \otimes T^{(0)}$.

5.1. 1-loop correction of $T_a^{(0)}$

The Feynman diagrams of the 1-loop corrections of T_a are shown in Fig. 3.

We denote the correction of the electric–magnetic (EM) vertex, the one shown in Fig. 3a, as $F^{(1)\text{EM}}$. To show the effect of the transverse momentum explicitly, we establish the longitudinal part and transverse part separately. The matrix element at tree level can be written as

$$\begin{aligned}
&F_a^{(0)} = F_{a\parallel}^{(0)} + F_{a\perp}^{(0)}, \qquad F_{a\parallel}^{(0)} = -e_q \bar{v}_{\bar{q}} \frac{\not{\varepsilon} \not{p}_\gamma}{2p_\gamma \cdot p_{\bar{q}}} P_L^\mu u_Q \\
&F_{a\perp}^{(0)} = e_q \bar{v}_{\bar{q}} \frac{2\varepsilon \cdot p_{\bar{q}}}{2p_\gamma \cdot p_{\bar{q}}} P_L^\mu u_Q
\end{aligned} \tag{20}$$

The transverse part is at order $O(\Lambda_{\text{QCD}}/m_Q)$. The corrections to each part is represented separately as

$$\begin{aligned}
F_a^{(1)\text{EM}} &= F_{a\parallel}^{(1)\text{EM}} + F_{a\perp}^{(1)\text{EM}} \\
F_{a\parallel}^{(1)\text{EM}} &= C_F g_s^2 \int \frac{d^d l}{(2\pi)^d} \bar{v}_{\bar{q}} i\gamma_\rho \frac{i(-\not{p}_{\bar{q}} - \not{l} + m_q)}{(p_{\bar{q}} + l)^2 - m_q^2} (-ie_q) \not{\varepsilon} \frac{i(-\not{p}_{\bar{q}} - \not{l} + \not{p}_\gamma + m_q)}{(p_{\bar{q}} + l - p_\gamma)^2 - m_q^2} \\
&\quad \times i\gamma^\rho \frac{i\not{p}_\gamma}{2p_{\bar{q}} \cdot p_\gamma} P_L^\mu \frac{-i}{l^2} u_Q \\
F_{a\perp}^{(1)\text{EM}} &= ie_q C_F g_s^2 \int \frac{d^d l}{(2\pi)^d} \bar{v}_{\bar{q}} \gamma_\rho \frac{(\not{p}_{\bar{q}} + \not{l} - m_q)}{(p_{\bar{q}} + l)^2 - m_q^2} \not{\varepsilon} \frac{(\not{p}_{\bar{q}} + \not{l} - \not{p}_\gamma - m_q)}{(p_{\bar{q}} + l - p_\gamma)^2 - m_q^2} \\
&\quad \times \gamma^\rho \frac{\not{p}_{\bar{q}}}{2p_{\bar{q}} \cdot p_\gamma} P_L^\mu \frac{1}{l^2} u_Q
\end{aligned} \tag{21}$$

After performing the momentum-integration, the result is

$$F_{a\parallel}^{(1)\mathrm{EM}} = F_{a\parallel}^{(0)}\frac{\alpha_s C_F}{4\pi}\left(N_{\mathrm{UV}} - \log\left(\frac{2p_\gamma\cdot p_{\bar{q}}}{\mu^2}\right) + 2\log\left(\frac{2p_\gamma\cdot p_{\bar{q}}}{m_q^2}\right)\right)$$
$$F_{a\perp}^{(1)\mathrm{EM}} = F_{a\perp}^{(0)}\frac{\alpha_s C_F}{4\pi}\left(N_{\mathrm{UV}} - \log\left(\frac{2p_{\bar{q}}\cdot p_\gamma}{\mu^2}\right) + 1\right) \tag{22}$$

where C_F is defined as $C_F = (N^2-1)/2N = 4/3$ for QCD. The transverse part $F_\perp^{(1)}$ is an order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$ contribution. Accordingly the 1-loop correction of the wave-function is $\Phi_q^{(1)} \otimes T_a^{(0)}$, which can also be written as $\Phi_q^{(1)} \otimes (T_{a\parallel}^{(0)} + T_{a\perp}^{(0)})$:

$$\Phi_q^{(1)} \otimes T_{a\parallel}^{(0)} = ie_q C_F g_s^2 \int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2}\bar{v}_{\bar{q}}\not{p}_\gamma \frac{(\not{l} + \not{p}_{\bar{q}} - m_q)}{(l+p_{\bar{q}})^2 - m_{\bar{q}}^2}\frac{2\not{p}_\gamma \not{\varepsilon}_\gamma^*}{4p_\gamma\cdot(p+l)p_\gamma\cdot p}P_L^\mu u_Q$$
$$\Phi_q^{(1)} \otimes T_{a\perp}^{(0)} = -ie_q g_s^2 C_F \int_0^1 d\alpha \int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2}$$
$$\times\ \bar{v}_{\bar{q}}\frac{4\not{\varepsilon}_\gamma(p_\gamma\cdot(p_{\bar{q}}+\alpha l)) - 4\not{p}_\gamma \varepsilon_\gamma\cdot(p_{\bar{q}}+\alpha l)}{(2p_\gamma\cdot(p_{\bar{q}}+\alpha l))^2}\frac{(\not{l}+\not{p}_{\bar{q}} - m_{\bar{q}})}{(l+p_{\bar{q}})^2 - m_{\bar{q}}^2}P_{L\mu}u_Q \tag{23}$$

$\Phi_q^{(1)} \otimes T_\perp^{(0)}$ is calculated in the light-cone coordinate (see also Appendix A in Ref. [7]), the result is

$$\Phi_q^{(1)} \otimes T_{a\parallel}^{(0)} = 2\frac{C_F\alpha_s}{4\pi}T_{a\parallel}^{(0)}\left(N_{\mathrm{UV}} - \log\frac{m_q^2}{\mu^2} + 2\right), \qquad \Phi_q^{(1)} \otimes T_{a\perp}^{(0)} = 0 \tag{24}$$

Using Eq. (12), we can obtain the hard-scattering kernel

$$T_a^{(1)\mathrm{EM}} = T_{a\parallel}^{(0)}\frac{C_F\alpha_s}{4\pi}\left(\log\frac{2k_{\bar{q}}\cdot p_\gamma}{\mu^2} - 4\right) + T_{a\perp}^{(0)}\frac{C_F\alpha_s}{4\pi}\left(-\log\frac{2k_{\bar{q}}\cdot p_\gamma}{\mu^2} + 1\right) \tag{25}$$

We find that, the hard-scattering kernel of the longitudinal direction is the same as the result in Ref. [8].

The correction of the weak vertex is shown in Fig. 3b. The result can be written as [20]

$$\begin{aligned}
&F_a^{(1)\mathrm{weak}} \\
&= -ie_q C_F g_s^2 \frac{i}{16\pi^2}\bar{v}_{\bar{q}}\not{\varepsilon}\left\{\frac{\not{p}_\gamma - \not{p}_{\bar{q}}}{-z}\left[\left(N_{\mathrm{UV}} - \log\frac{y}{\mu^2} + y_1 - 2\log\frac{xz}{y^2} + \frac{x}{x-y}\log\frac{x}{y}\right)\right.\right. \\
&\quad + \frac{wy}{(x-y)^2y^2}\left(x^2\left(-4\log\frac{xz}{y^2} - 2\right) + xy\left(5\log\frac{x}{y} + 8\log\frac{z}{y} + 5\right)\right. \\
&\quad \left. + y^2\left(2\log\frac{y}{x} + 4\log\frac{y}{z} - 3\right)\right) + \frac{x^3 z}{(x-y)^2y^2}\left(-3y_1 + 8\log\frac{xz}{y^2} - 4\right) \\
&\quad + \frac{x^2yz}{(x-y)^2y^2}\left(7y_1 + 13\log\frac{y}{x} + 19\log\frac{y}{z} + 14\right) \\
&\quad + \frac{xy^2z}{(x-y)^2y^2}\left(-5y_1 + 4\log\frac{x}{y} + 14\log\frac{z}{y} - 17\right) \\
&\quad \left. + \frac{y^3z}{(x-y)^2y^2}\left(y_1 - 3\log\frac{z}{y} + 2\log\frac{x}{y} + 7\right)\right]P_L^\mu \\
&\quad \left. + \frac{\not{p}_\gamma - \not{p}_{\bar{q}}}{-z}4p_{\bar{q}}^\mu \not{p}_Q(1-\gamma_5)\frac{\log\frac{x}{y}}{2(x-y)} - 2m_Q P_R^\mu \frac{1}{y}\log\frac{xz}{y^2}\right\}u_Q
\end{aligned} \tag{26}$$

with

$$y_1 = -\pi^2 + 2\,\mathrm{Li}_2\left(1 - \frac{x}{y}\right) - \log^2 \frac{xz}{y^2} + \log^2 \frac{x}{y} \tag{27}$$

In the expressions above, we have already neglected the terms which are irrelevant for the discussion, because they will not contribute to the matrix element up to order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ when convoluted with the distribution amplitudes, due to their Dirac structures [8]. In the rest of this section, we will always use this simplification if possible.

The correction of wave-function is $\Phi_Q^{(1)} \otimes T_a^{(0)}$, which can be written as

$$\begin{aligned}
&\Phi_Q^{(1)} \otimes T_{a\parallel}^{(0)} \\
&\quad = -2ie_q C_F g_s^2 \bar{v}_{\bar{q}} \not{\epsilon} \frac{\not{p}_\gamma}{2p_{\bar{q}} \cdot p_\gamma} P_L^\mu \frac{i}{16\pi^2} \Bigg\{ \left(\left(\log \frac{z}{y} - 1 \right) \left(1 - \frac{z}{y} \right) + \frac{m_Q \not{p}_\gamma}{y} \right) N_{\mathrm{UV}} \\
&\qquad - \left(\left(2 - \frac{\pi^2}{3} + \left(\frac{1}{4} \log^2 \frac{x}{\mu^2} - \log^2 \frac{y\mu}{zm_Q} \right) \right) \left(1 + \frac{z}{y} \right) \right. \\
&\qquad \left. + \left(\frac{z}{y} - 1 \right) \log \frac{x}{\mu^2} \right) - \left(\left(\frac{m_Q \not{p}_\gamma}{y} \right) \left(\log \frac{x}{\mu^2} - 2 \right) \right) \Bigg\} u_Q
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\Phi_Q^{(1)} \otimes T_{a\perp}^{(0)} = {}& -2ie_q C_F g_s^2 \frac{i}{16\pi^2} \bar{v}_{\bar{q}} P_L^\mu \Bigg\{ \left(-\frac{m_Q}{y} \left(\log \frac{z}{y} + 1 \right) \right) N_{\mathrm{UV}} \\
& - \frac{m_Q}{y} \left(\left(\log \frac{x}{\mu^2} - 2 \right) \log \frac{y}{z} - \log \frac{x}{\mu^2} + 4 \right) \Bigg\} \not{\epsilon} u_Q
\end{aligned} \tag{29}$$

With Eq. (6) and Eq. (12), the hard-scattering kernel can be obtained through the following equation

$$\Phi^{(0)} \otimes T_a^{(1)\mathrm{weak}} = F_a^{(1)\mathrm{weak}} - \left(\Phi^{(1)} \otimes T_a^{(0)} \right) \tag{30}$$

The correction in Fig. 3c is

$$F_a^{(1)\mathrm{wfc}} = -\frac{C_F \alpha_s}{4\pi} F_a^{(0)} \left(N_{\mathrm{UV}} - \log \left(\frac{2p_\gamma \cdot p_{\bar{q}}}{\mu^2} \right) + 1 \right) \tag{31}$$

We find that the correction of the wave-function also vanishes as Ref. [8], i.e.

$$\Phi_{\mathrm{Wfc}}^{(1)} \otimes T_a^{(0)} = 0 \tag{32}$$

Then we obtain

$$T^{\mathrm{wfc}(1)} = \frac{\alpha_s C_F}{4\pi} \left(T_{a\parallel}^{(0)} + T_{a\perp}^{(0)} \right) \left(\log \frac{2p_\gamma \cdot k_{\bar{q}}}{\mu^2} - 1 \right) \tag{33}$$

The corrections of the external legs and the box diagram are equal to the relevant corrections to the wave-functions because we use the renormalization scale equal to the factorization scale, so

$$T_a^{(1)\mathrm{extq}} = T_a^{(1)\mathrm{extQ}} = T_a^{(1)\mathrm{box}} = 0 \tag{34}$$

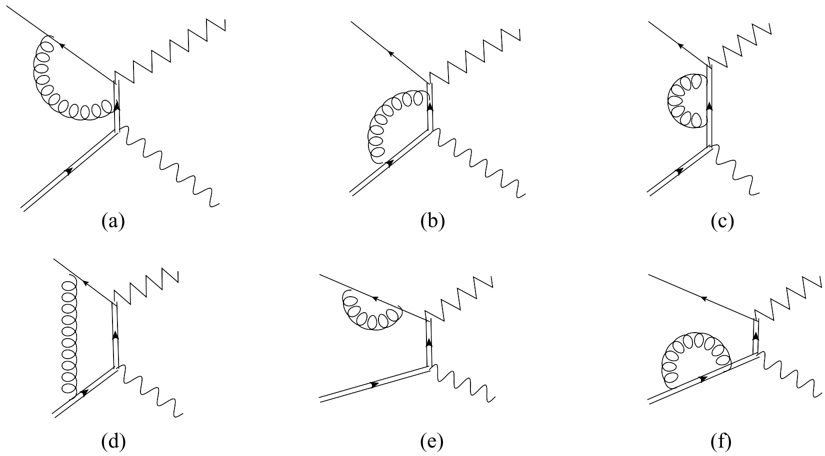

Fig. 4. 1-loop QCD correction of F_b.

5.2. 1-loop correction of $T_b^{(0)}$

The corrections of T_b are also order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contributions as the tree level, the Feynman diagrams of the 1-loop corrections of T_b are shown in Fig. 4.

The correction of the weak vertex is

$$F_b^{(1)\mathrm{weak}} = -ie_Q C_F g_s^2 \frac{i}{16\pi^2}\bar{v}_{\bar{q}}\left\{\left[N_{\mathrm{UV}} - \log\frac{y}{\mu^2} - 2 + \frac{x}{y-x} + \frac{x^2}{(x-y)^2}\log\frac{x}{y} + \frac{2wyy_2}{w-z} + \frac{wy\log\frac{x}{y}(3\log\frac{x}{x-y} + \log\frac{y}{x-y})}{(w-z)^2}\right]P_L^\mu \frac{\not{p}_\gamma}{y} + 2P_L^\mu \not{p}_q \frac{2yy_2(w-z) + y(\log\frac{y}{x}(3\log\frac{x}{x-y} + \log\frac{y}{x-y}))}{2(w-z)^2}\right\}\not{\epsilon}u_Q \tag{35}$$

with

$$y_2 = \frac{1}{2(w-z)}\left(\log\left(\left(\frac{x}{w+x-y-z}\right)^3 \frac{y-w+z}{x-y+w-z}\right)\log\left(\frac{x}{y-w+z}\right)\right) \tag{36}$$

and the correction of the wave-function is

$$\Phi_q^{(1)} \otimes T_b^{(0)} \sim O\left(\frac{1}{m_Q^2}\right) \tag{37}$$

the hard kernel can be obtained using

$$\Phi^{(0)} \otimes T_b^{\mathrm{weak}(1)} = F_b^{\mathrm{weak}(1)} \tag{38}$$

The results of the EM vertex is

$$F_b^{(1)\mathrm{EM}} = -ie_Q C_F g_s^2 \bar{v}_{\bar{q}} \frac{i}{16\pi^2} P_L^\mu \left\{\frac{m_Q\not{\epsilon}}{y}\frac{-2y}{x-y}\log\frac{x}{y} + \frac{\not{p}_\gamma\not{\epsilon}}{y}\left(2N_{\mathrm{UV}} - \log\frac{x}{\mu^2} - \frac{x}{y}\left(-2\,\mathrm{Li}_2\left(1-\frac{y}{x}\right) + \frac{\pi^2}{3}\right) - \frac{6x-y}{x-y}\log\frac{x}{y}\right)\right\}u_Q$$

$$
\begin{aligned}
\Phi_Q^{(1)} \otimes T_b^{(0)} &= -2ie_Q C_F g_s^2 \frac{i}{16\pi^2} \bar{v}_{\bar{q}} P_L^\mu \frac{\not{p}_\gamma \not{s}}{y} u_Q \left(N_{\mathrm{UV}} - \log\frac{x}{\mu^2} + 2 \right) \\
T_b^{(1)\mathrm{EM}} &= T_b^{(0)} \frac{C_F g_s^2}{4\pi} \left(\log\frac{m_Q^2}{\mu^2} - 4 - \frac{m_Q^2}{2p_\gamma \cdot k_Q} \left(-2\,\mathrm{Li}_2\left(1 - \frac{2p_\gamma \cdot k_Q}{m_Q^2}\right) + \frac{\pi^2}{3} \right) \right. \\
&\quad \left. - \frac{6m_Q^2 - 2p_\gamma \cdot k_Q}{m_Q^2 - 2p_\gamma \cdot k_Q} \log\frac{m_Q^2}{2p_\gamma \cdot k_Q} \right) \\
&\quad + ie_Q C_F g_s^2 \frac{i}{16\pi^2} \bar{v}_{\bar{q}} \frac{P_L^\mu \not{s} m_Q}{2p_\gamma \cdot k_Q} u_Q \times \frac{4p_\gamma \cdot k_Q}{m_Q^2 - 2p_\gamma \cdot k_Q} \log\frac{m_Q^2}{2p_\gamma \cdot k_Q}
\end{aligned} \tag{39}
$$

The results in Fig. 4c is

$$
\begin{aligned}
F_b^{(1)\mathrm{wfc}} &= -ie_Q C_f g_s^2 \frac{i}{16\pi^2} \bar{v}_{\bar{q}} P_L^\mu \left\{ \left(-N_{\mathrm{UV}} - 1 + \log\frac{x}{\mu^2} - \frac{x}{(x-y)} + \frac{y(2x-y)}{(x-y)^2} \log\frac{x}{y} \right. \right. \\
&\quad \left. + \frac{2x}{y} \left(-3N_{\mathrm{UV}} - 5 + 3\log\frac{x}{\mu^2} + \frac{x}{x-y} + \frac{(2x-3y)y}{(x-y)^2} \log\frac{x}{y} \right) \right) \frac{\not{p}_\gamma}{y} \not{s} u_Q \\
&\quad \left. + \left(-3N_{\mathrm{UV}} - 5 + 3\log\frac{x}{\mu^2} + \frac{x}{x-y} + \frac{(2x-3y)y}{(x-y)^2} \log\frac{x}{y} \right) \frac{m_Q \not{s}}{y} u_Q \right\} \\
\Phi_{\mathrm{Wfc}}^{(1)} \otimes T_b^{(0)} &= 0 \\
T_b^{(1)\mathrm{wfc}} &= \frac{\alpha_s C_F}{4\pi} T_{b\parallel}^{(0)} \left(-1 + \log\frac{x}{\mu^2} - \frac{x}{x-y} + \frac{y(2x-y)}{(x-y)^2} \log\frac{x}{y} \right. \\
&\quad \left. + \frac{2x}{y} \left(-5 + 3\log\frac{x}{\mu^2} + \frac{x}{x-y} + \frac{(2x-3y)y}{(x-y)^2} \log\frac{x}{y} \right) \right) \\
&\quad - ie_Q C_F g_s^2 \frac{i}{16\pi^2} \bar{v}_{\bar{q}} P_L^\mu \not{s} u_Q \\
&\quad \times \left(\frac{m_Q}{y} \left(-5 + 3\log\frac{x}{\mu^2} + \frac{x}{x-y} + \frac{y(2x-3y)}{(x-y)^2} \log\frac{x}{y} \right) \right) \Bigg|_{p_Q \to k_Q}
\end{aligned} \tag{40}
$$

And the correction of the external legs and the box correction are also equal to each other, so we also have

$$
T_b^{(1)\mathrm{extq}} = T_b^{(1)\mathrm{extQ}} = T_b^{(1)\mathrm{box}} = 0 \tag{41}
$$

5.3. 1-loop correction of $T_c^{(0)}$

The correction of the distribution function with gluons in Wilson line does not have correspondent 1-loop QCD corrections. And because the momentum of light quark and heavy quark show up together as $k_Q + k_{\bar{q}}$ in $T_c^{(0)}$, we find

$$
\left(\frac{\partial}{\partial k_Q} - \frac{\partial}{\partial k_{\bar{q}}} \right) T_c^{(0)} = 0 \tag{42}
$$

and we obtain

$$
T_c^{(1)q} = T_c^{(1)Q} = T_c^{(1)\mathrm{wfc}} = 0 \tag{43}
$$

All the other corrections are similar as Eq. (34) and Eq. (41), we find

$$T_c^{(1)\mathrm{extq}} = T_c^{(1)\mathrm{extQ}} = T_c^{(1)\mathrm{trangle}} = 0 \tag{44}$$

5.4. 1-loop result summery

With Eqs. (25), (30), (33), (34), (38)–(41), (43) and (44), we can establish the order α_s hard-scattering kernel. We find that, $T^{(1)}$ is IR-finite up to order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$, so the factorization is proved up to the order of $O(1/m_Q)$ corrections.

It is well known that, the factorization will fail with the light mesons. However, we find that the $\Lambda_{\mathrm{QCD}}/m_Q$ expansion is irrelevant by briefly investigating the factorization at higher orders of $O(\Lambda_{\mathrm{QCD}}/m_Q)$.

In the calculations above, we find that the IR divergences in the corrections of the external legs and the box diagrams are cancelled exactly, while the remaining IR divergences are all collinear divergences which show up in the amplitudes with one of the vertexes of the gluon propagator connecting to the external light quark, which are denoted in Fig. 3a and Fig. 4a. In $T_a^{(1)\mathrm{EM}}$, there are no neglected $O(\Lambda_{\mathrm{QCD}}/m_Q)^2$ or higher order contributions. As a result, the IR divergences at higher orders only survive in $T_b^{(1)\mathrm{weak}}$. We also find that, there are both higher order IR divergences in $F_b^{(1)\mathrm{weak}}$ and $\Phi_q^{(1)} \otimes T_b^{(0)}$. We shall investigate whether those IR divergences can be cancelled.

We concentrate on the IR region of the loop integration, using $l_\epsilon \to 0$, we find

$$\begin{aligned} &F_{b\,\mathrm{IR}}^{\mathrm{weak}(1)} \\ &= \lim_{l_\epsilon \to 0}\Bigg(-ie_Q C_F g_s \int_{-l_\epsilon}^{l_\epsilon} \frac{d^d l}{(2\pi)^d} \bar{v}_q \frac{1}{l^2}\gamma_\rho \frac{-\not{p}_{\bar{q}}}{(l+p_{\bar{q}})^2 - m_q^2} P_L^\mu \\ &\quad \times \frac{2p_Q^\rho \not{p}_Q + 2p_\gamma^\rho \not{p}_\gamma - 2p_Q^\rho \not{p}_\gamma - 2p_\gamma^\rho \not{p}_Q + 2\gamma^\rho p_Q\cdot p_\gamma + 2(p_Q - p_\gamma)^\rho m_Q}{(2p_Q\cdot p_\gamma)^2} \not{\epsilon} u_Q\Bigg) \end{aligned} \tag{45}$$

We also bring back the neglected higher order terms in the numerator of $T_b^{(0)}$, and the correction of the wave-function is

$$\begin{aligned} &T_b^{(0)} = e_Q P_L^\mu \frac{\not{k}_Q - \not{p}_\gamma + m_Q}{(k_Q - p_\gamma)^2 - m_Q^2}\not{\epsilon} \\ &\left(\Phi_q^{(1)} \otimes T^{(0)}\right)_{\mathrm{IR}} = \lim_{l_\epsilon\to 0}\Bigg(-ie_Q C_F g_s^2 \int_{-l_\epsilon}^{l_\epsilon} \frac{d^d l}{(2\pi)^d}\frac{1}{l^2}\bar{v}_q \gamma^\rho \frac{\not{p}_{\bar{q}}}{(l+p_{\bar{q}})^2 - m_q^2} P_L^\mu \\ &\qquad \times \frac{-\gamma_\rho 2p_Q\cdot p_\gamma - 2(p_Q - p_\gamma)^\rho(\not{p}_Q - \not{p}_\gamma + m_Q)}{(2p_Q\cdot p_\gamma)^2}\not{\epsilon} u_Q\Bigg) \end{aligned} \tag{46}$$

we find

$$F_{b\,\mathrm{IR}}^{(1)} - \left(\Phi^{(1)} \otimes T^{(0)}\right)_{\mathrm{IR}} = 0 \tag{47}$$

This result indicates that, the factorization is valid at any order of $O(\Lambda_{\mathrm{QCD}}/m_Q)$, as a result, the valid region of factorization in Ref. [8] is extended. The failure of the factorization approach in the light meson decays is due to the bad convergence behaviour in the $O(\alpha_s)$ expansion.

We then concentrate on the contribution of the hard scattering kernel to the amplitude. The amplitude can be obtained by replacing the wave-function with the one obtained in Ref. [16]

$$\Phi^{(0)}(k_q, k_Q) = \frac{1}{\sqrt{3}} \int d^3k \Psi(k) \frac{1}{\sqrt{2}} M|0\rangle \\ \times \delta^3(\vec{k}_{\bar{q}} + \vec{k})\delta^3(\vec{k}_Q - \vec{k})\delta\Big(k_{\bar{q}0} - \sqrt{k^2 + m_q^2}\Big)\delta\Big(k_{Q0} - \sqrt{k^2 + m_Q^2}\Big) \quad (48)$$

with

$$M = \sum_i b_Q^{i+}(\vec{k}, \uparrow)d_q^{i+}(-\vec{k}, \downarrow) - b_Q^{i+}(\vec{k}, \downarrow)d_q^{i+}(-\vec{k}, \uparrow), \\ \Psi(\vec{k}) = 4\pi\sqrt{m_P \lambda_P^3} e^{-\lambda_P |\vec{k}|} \quad (49)$$

The matrix element can be written as

$$F^\mu(\mu) \\ = \frac{1}{(2\pi)^3} \frac{3}{\sqrt{6}} \int d^3k \int d^4k_Q \int d^4k_{\bar{q}} \Psi(k) \operatorname{Tr}\big[M \cdot \big(T^{(0)\mu}(k_{\bar{q}}, k_Q) + T^{(1)\mu}(k_{\bar{q}}, k_Q)\big)\big] \\ \times \delta^3(\vec{k}_{\bar{q}} + \vec{k})\delta^3(\vec{k}_Q - \vec{k})\delta\Big(k_{\bar{q}0} - \sqrt{k^2 + m_q^2}\Big)\delta\Big(k_{Q0} - \sqrt{k^2 + m_Q^2}\Big) \quad (50)$$

After convolution with the wave-functions of the heavy mesons, some of the terms will have identical contributions to the matrix element up to order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ due to their Dirac structures. As a result, we find that the F^μ can be simplified as a function of four different types of Dirac structures, and can be written as

$$F^\mu(\mu) = \sum_n \frac{1}{(2\pi)^3} \frac{3}{\sqrt{6}} \int d^3k \Psi(k) \operatorname{Tr}\big[C_n(p_Q, p_{\bar{q}}, \mu) M \cdot K_n(p_Q, p_{\bar{q}})\big] \quad (51)$$

with $p_q = (\sqrt{m_q^2 + k^2}, -\vec{k})$ and $p_Q = (\sqrt{m_Q^2 + k^2}, \vec{k})$ denote the on-shell momenta of the light anti-quark and the heavy quark in the bound state. And the K_n are defined as

$$K_1(k_Q, k_{\bar{q}}) = T_{a\parallel}^{(0)}, \qquad K_2(k_Q, k_{\bar{q}}) = T_{a\perp}^{(0)} \\ K_3(k_Q, k_{\bar{q}}) = T_b^{(0)}, \qquad K_4(k_Q, k_{\bar{q}}) = e_Q \frac{P_L^\mu \not{\epsilon} m_Q}{2p_\gamma \cdot k_Q} \quad (52)$$

Except for $C_1 K_1$, all the other products are contribution of order $O(\Lambda_{\mathrm{QCD}}/m_Q)$, for clarity, we define $C_1 = C_1^0 + C_1^1$, with C_1^m represents order $O(\Lambda_{\mathrm{QCD}}/m_Q)^m$ contribution, the coefficients are

$$C_1^0(p_q, p_Q, \mu) = 1 + \frac{\alpha_s C_F}{4\pi}\bigg(-\log\frac{y}{\mu^2} + y_1 - 2\log\frac{xz}{y^2} + \frac{x}{x-y}\log\frac{x}{y} - 4 + \frac{2\pi^2}{3} \\ + 2\log^2\frac{y}{z} - 2\log\frac{y}{z}\log\frac{x}{\mu^2} + 2\log\frac{x}{\mu^2} + 2\log\frac{z}{\mu^2} - 5\bigg) \quad (53)$$

$$
\begin{aligned}
C_1^1(p_q, p_Q, \mu) = {} & \frac{w}{(x-y)^2 y}\left(x^2\left(-4\log\frac{xz}{y^2}-2\right)+xy\left(5\log\frac{x}{y}+8\log\frac{z}{y}+5\right)\right.\\
& \left.+y^2\left(2\log\frac{y}{x}+4\log\frac{y}{z}-3\right)\right)+\frac{x^3 z}{(x-y)^2y^2}\left(-3y_1+8\log\frac{xz}{y^2}-4\right)\\
& +\frac{x^2 z}{(x-y)^2 y}\left(7y_1+13\log\frac{y}{x}+19\log\frac{y}{z}+14\right)\\
& +\frac{xz}{(x-y)^2}\left(-5y_1+4\log\frac{x}{y}+14\log\frac{z}{y}-17\right)\\
& +\frac{yz}{(x-y)^2}\left(y_1-3\log\frac{z}{y}+2\log\frac{x}{y}+7\right)\\
& -\frac{w}{2(x-y)}\log\frac{x}{y}+\left(\frac{2w}{y}-\frac{4xz}{y^2}\right)\log\frac{xz}{y^2}\\
& +\left(\frac{2w}{y}-\frac{4xz}{y^2}\right)\left(\log\frac{x}{\mu^2}\log\frac{y}{z}-2\log\frac{y}{z}-\log\frac{x}{\mu^2}+4\right)\\
& -\left(2-\frac{\pi^2}{3}-\log^2\frac{y}{z}+\log\frac{y}{z}\log\frac{x}{\mu^2}+\log\frac{x}{\mu^2}\right)\frac{2z}{y}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
C_2(p_q, p_Q, \mu) = 1 + {} & \frac{\alpha_s C_F}{4\pi}\left(-\log\frac{y}{\mu^2}+y_1-2\log\frac{xz}{y^2}+\log\frac{x}{y}+\frac{xz}{yw}\log\frac{xz}{y^2}\right.\\
& \left.-\frac{4zx}{yw}\left(\log\frac{x}{\mu^2}\log\frac{y}{z}-2\log\frac{y}{z}-\log\frac{x}{\mu^2}+4\right)\right)
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
& C_3(p_q, p_Q, \mu)\\
& = 1 + \frac{\alpha_s C_F}{4\pi}\left(\left(\log\frac{x}{\mu^2}-2-\frac{2x}{y}\left(-\mathrm{Li}_2\left(1-\frac{y}{x}\right)+\frac{\pi^2}{3}\right)-\frac{6x-y}{x-y}\log\frac{x}{y}\right)\right.\\
& \quad -1+\log\frac{x}{\mu^2}-\frac{x}{x-y}+\frac{y(2x-y)}{(x-y)^2}\log\frac{x}{y}\\
& \quad +\frac{2x}{y}\left(-5+3\log\frac{x}{\mu^2}+\frac{x}{x-y}+\frac{(2x-3y)y}{(x-y)^2}\log\frac{x}{y}\right)-\log\frac{y}{\mu^2}-2+\frac{x}{y-x}\\
& \quad \left.+\frac{x^2}{(x-y)^2}\log\frac{x}{y}+\frac{2wyy_2}{w-z}+\frac{wy\log\frac{x}{y}(3\log\frac{x}{x-y}+\log\frac{y}{x-y})}{(w-z)^2}\right)
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
& C_4(p_q, p_Q, \mu)\\
& = \frac{\alpha_s C_F}{4\pi}\left(\left(\frac{2xz^2}{yw}-2z\right)\frac{2y_2 y(w-z)+\log\frac{y}{x}(3y\log\frac{x}{x-y}+y\log\frac{y}{x-y})}{(w-z)^2}\right.\\
& \quad \left.-\frac{2y}{x-y}\log\frac{x}{y}+\left(-5+3\log\frac{x}{\mu^2}+\frac{x}{x-y}+\frac{y(2x-3y)}{(x-y)^2}\log\frac{x}{y}\right)\right)
\end{aligned}
\tag{57}
$$

with x, y, z and w defined in Eq. (18), and y_1, y_2 defined in Eq. (27) and Eq. (36).

6. Large logarithm resummation

In the expression of F^{μ}, large logarithms show up. Those large logarithms need to be resummed so that the result is phenomenologically reliable. We concentrate on the large logarithms at order $O(\Lambda_{\mathrm{QCD}}/m_Q)^0$, because terms like $(\Lambda_{\mathrm{QCD}} \log \frac{\sqrt{m_Q \Lambda_{\mathrm{QCD}}}}{m_Q})/m_Q$ are suppressed and not large. In this section, we use $y = 2p_\gamma \cdot p_Q = 2E_\gamma m_Q$ for simplicity. We concentrate on C_1^0, which can also be written as

$$\begin{aligned}
&C_1^0(p_q, p_Q, \mu) \\
&\quad = 1 + \frac{\alpha_s(\mu) C_F}{4\pi}\left(-2\,\mathrm{Li}_2\left(1 - \frac{y}{x}\right) - 2\log^2\frac{x}{y} - \frac{y}{x-y}\log\frac{y}{x} + 2\log\frac{y}{x}\right. \\
&\qquad \left. + 2\log\frac{x}{y}\log\frac{x}{\mu^2} + 3\log\frac{x}{\mu^2} - \log^2\frac{x}{\mu^2} - 6 - \frac{\pi^2}{12} + \log^2\frac{\mu^2}{z} - 3 - \frac{\pi^2}{4}\right)
\end{aligned} \tag{58}$$

When $\mu \sim m_Q$, the large logarithms are the same as Ref. [8]. However, when μ is set to $\mu \sim \sqrt{m_Q \Lambda_{\mathrm{QCD}}}$, the large logarithms are different from Ref. [8] by

$$-\frac{1}{2}\log^2\frac{m_Q}{\mu^2} + \frac{1}{2}\log\frac{m_Q}{\mu^2} \tag{59}$$

This is because the corrections of wave-functions in this work are not calculated in HQET framework as Ref. [8], in which m_Q will not appear in the result.

6.1. Renormalization group equation (RGE) evolution

We start with the RGE at order $O(\Lambda_{\mathrm{QCD}}/m_Q)^0$, and use the method introduced in Ref. [13]. The RGE of C_1^0 is

$$\mu\frac{\partial}{\partial\mu}C_1^0(\mu) = \gamma(\mu) C_1^0(\mu) \tag{60}$$

with $\gamma(\mu)$ defined as anomalous dimension of C_1^0, which can be obtained by using the counter terms of $C_1^0 K_1$. The counter terms at the order $O(\Lambda_{\mathrm{QCD}}/m_Q)^0$ can be written as

$$Z = \frac{\alpha_s(\mu) C_F}{4\pi}\frac{2}{\epsilon}\left(2\log\frac{z}{y} - 3\right) \tag{61}$$

We can use the counter terms and the β function in QCD to calculate $\gamma(\mu)$

$$\beta = -g\frac{\epsilon}{2} + O\big(g^3\big), \qquad \gamma(\mu) = Z^{-1}\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)Z \tag{62}$$

The result is

$$\gamma(\mu) = -\frac{\alpha_s(\mu) C_F}{2\pi}\left(2\log\frac{\mu^2}{y} + 2\log\frac{z}{\mu^2} - 3\right) \tag{63}$$

With this result, we can solve the RGE of the coefficient. Assume the coefficient can be written as a hard function multiplied by a jet function [9,21], so

$$\begin{aligned}
&C(\mu) = H(\mu) J(\mu), \qquad \gamma(\mu) = \gamma_H(\mu) + \gamma_J(\mu) \\
&\left(\mu\frac{\partial}{\partial\mu}H(\mu)\right)J(\mu) + H(\mu)\left(\mu\frac{\partial}{\partial\mu}J(\mu)\right) = \gamma_H(\mu) H(\mu) J(\mu) + \gamma_J(\mu) H(\mu) J(\mu)
\end{aligned} \tag{64}$$

The RGE of the hard function is

$$\mu\frac{\partial}{\partial\mu}H(\mu)=\gamma_H(\mu)H(\mu) \tag{65}$$

By splitting the coefficient $C_1^{(0)}$ into H and J, we assume that the natural scale of H is m_Q, while the natural scale of J is $\sqrt{m_Q\Lambda_{\mathrm{QCD}}}$, which is also the case for $\gamma(\mu)$ with a $\log(\mu^2/y)$ term and a $\log(\mu^2/z)$ term. So we split the $\gamma(\mu)$ such that $\gamma_H(\mu)$ is the sum of the $\log(\mu^2/y)$ term and an undetermined constant n', so

$$\gamma_H(\mu)=-\frac{\alpha_s(\mu)C_F}{2\pi}\left(2\log\frac{\mu^2}{y}+n'\right) \tag{66}$$

Similar as Refs. [8,14], we find

$$\begin{aligned}\gamma_{H_{\mathrm{LO}}}&=-\frac{\alpha_s(\mu)C_F}{\pi}2\log\frac{\mu}{m_Q}\\ \gamma_{H_{\mathrm{NLO}}}&=-\frac{\alpha_s(\mu)C_F}{2\pi}\left(n'-2\log\frac{2E_\gamma}{m_Q}\right)-2C_FB\frac{\alpha_s^2(\mu)}{(2\pi)^2}\log\frac{\mu}{m_Q}\end{aligned} \tag{67}$$

The $\gamma_{H_{\mathrm{NLO}}}$ is the same as Refs. [8,14], so the solution is

$$\begin{aligned}H(\mu)&=\exp\left(\frac{f_0}{\alpha_s(m_Q)}+f_1\right)H(m_Q)\\ f_0&=\alpha_s(m_Q)\left(-2\frac{4\pi C_F}{\beta_0^2\alpha_s(m_Q)}\left(\frac{1}{r}-1+\log r\right)\right)\\ f_1&=-\frac{C_F\beta_1}{\beta_0^3}\left(1-r+r\log r-\frac{1}{2}\log^2 r\right)+\frac{C_F}{\beta_0}\left(n'-2\log\frac{y}{x}\right)\log r\\ &\quad-\frac{2C_FB}{\beta_0^2}(r-1-\log r)\end{aligned} \tag{68}$$

with $r=\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}$, $\beta_0=\frac{11C_A}{3}-\frac{2N_f}{3}$ and $\beta_1=\frac{34C_A^2}{3}-\frac{10C_AN_f}{3}-2C_FN_f$, where $C_A=3$ for QCD, N_f is the number of the flavour of quarks taken into account, and B can only been derived from 2-loop calculations. In Ref. [14], by comparing the result with $B\to X_s\gamma$ and $B\to X_ul\bar{\nu}$ in Ref. [22], B is found to be $B=C_A(\frac{67}{18}-\frac{\pi^2}{6})-\frac{5N_f}{9}$. So the result of H is

$$\begin{aligned}H(\mu)=H(m_Q)\exp\biggl(&\frac{\alpha_s(m_Q)C_f}{4\pi}\left(-4\log^2\frac{\mu}{m_Q}+4\log\frac{y}{x}\log\frac{\mu}{m_Q}-2n'\log\frac{\mu}{m_Q}\right)\\ &+O\left(\alpha_s^2\right)\biggr)\end{aligned} \tag{69}$$

6.2. The resummation

The hard function can be derived by using the method introduced in Ref. [9]. With x_γ defined as $x_\gamma=2E_\gamma/m_Q$, the result is

$$\begin{aligned}&\hat{H}\left(\frac{2E_\gamma}{\mu}\right)\\ &\quad=1+\frac{\alpha_s(\mu)C_F}{4\pi}\left(-2\,\mathrm{Li}_2(1-x_\gamma)-2\log^2\frac{2E_\gamma}{\mu}-\frac{1}{1-x_\gamma}\log x_\gamma+2\log\frac{2E_\gamma}{\mu}\right)\end{aligned} \tag{70}$$

and

$$H(m_Q) = \hat{H}\left(\frac{2E_\gamma}{m_Q}\right) \tag{71}$$

we also find

$$H(m_Q) = C_{3,6}^{\text{SCET}}(m_Q) \tag{72}$$

With $n' = 3$, the evaluation of the RGE will correctly resum the large logarithms, which can be shown explicitly by expanding the solution of RGE of hard function

$$\begin{aligned}
H(\mu) &= H(m_Q)\left(1 + \frac{\alpha_s(m_Q)C_F}{4\pi}\left(-4\log^2\frac{\mu}{m_Q} + 4\log\frac{y}{x}\log\frac{\mu}{m_Q} - 6\log\frac{\mu}{m_Q}\right)\right) \\
&\quad + O\left(\alpha_s^2\right) \\
&= 1 + \frac{\alpha_s(m_Q)C_F}{4\pi}\left(-2\,\text{Li}_2\left(1-\frac{y}{x}\right) - 2\log^2\frac{x}{y} - \frac{y}{x-y}\log\frac{y}{x} + 2\log\frac{y}{x}\right. \\
&\quad \left. + 2\log\frac{x}{y}\log\frac{x}{\mu^2} + 3\log\frac{x}{\mu^2} - \log^2\frac{x}{\mu^2} - 6 - \frac{\pi^2}{12}\right) + O\left(\alpha_s^2\right)
\end{aligned} \tag{73}$$

Comparing Eqs. (60) and (64) with Eq. (73), we find that there are no more large logarithms in the remaining terms, which gives the jet function

$$J(\mu) = 1 + \frac{\alpha_s(\mu)C_F}{4\pi}\left(\log^2\frac{\mu^2}{z} - 3 - \frac{\pi^2}{4}\right) \tag{74}$$

The $\gamma_J(\mu)$ can been obtained by subtracting $\gamma_H(\mu)$ from $\gamma(\mu)$

$$\gamma_J(\mu) = \frac{\alpha_s(\mu)C_F}{\pi} 2\log\frac{\mu}{\sqrt{z}} \tag{75}$$

The solution of RGE of jet function is

$$J(\mu) = J(\sqrt{z}) \times \exp\left(\frac{\alpha_s(\sqrt{z})C_F}{4\pi}\log^2\frac{\mu^2}{z^2} + O\left(\alpha_s^2\right)\right) \tag{76}$$

We find that, Eq. (76) can also correctly resum the large logarithms which will show up in jet function when μ is evaluated to lower than $\sqrt{\Lambda_{\text{QCD}} m_Q}$.

When $\mu > \sqrt{\Lambda_{\text{QCD}} m_Q}$, the resummed result at order $O(\alpha_s(\Lambda_{\text{QCD}}/m_Q)^0)$ is

$$\begin{aligned}
C_1^0 &= \left\{1 + \frac{\alpha_s(m_Q)C_F}{4\pi}\left(-2\,\text{Li}_2\left(1-\frac{y}{x}\right) - 2\log^2\frac{y}{x}\right.\right. \\
&\quad \left.\left. - \frac{y}{x-y}\log\frac{y}{x} + 2\log\frac{y}{x} - 6 - \frac{\pi^2}{12}\right)\right\} \\
&\quad \times \exp\left(\frac{\alpha_s(m_Q)C_f}{4\pi}\left(-4\log^2\frac{\mu}{m_Q} + 4\log\frac{y}{x}\log\frac{\mu}{m_Q} - 6\log\frac{\mu}{m_Q}\right)\right) \\
&\quad \times \left(1 + \frac{\alpha_s(\mu)C_f}{4\pi}\left(\log^2\frac{\mu^2}{z} - 3 - \frac{\pi^2}{4}\right)\right)
\end{aligned} \tag{77}$$

At the leading order of $O(\Lambda_{\text{QCD}}/m_Q)$, z can be written as $z = 2p_\gamma \cdot k_{\bar{q}} = \sqrt{2}k_+ E_\gamma$. Using $\Lambda_{\text{QCD}} = 200$ MeV, $m_b = 4.98$ GeV, and $k_+ = \Lambda_{\text{QCD}}$, $E_\gamma = \frac{m_Q}{4}$, we can show the evaluation of coefficient C_1^0 in Fig. 5.

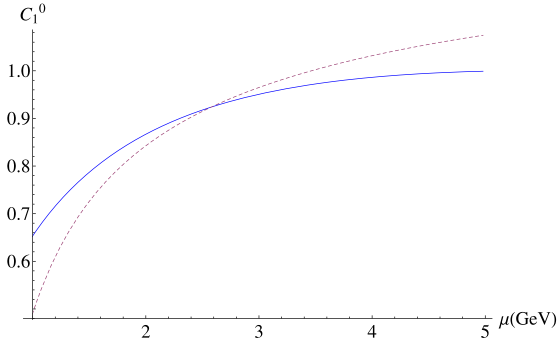

Fig. 5. C_1^0 as function of μ, with $E_\gamma = \frac{m_Q}{4}$, $k_+ = \Lambda_{\mathrm{QCD}}$. The solid line is the resummed result, and the dotted line is the un-resummed one.

7. Numerical applications

The amplitude is derived in Eq. (50), which can be decomposed as [6,23]

$$\langle\gamma|\bar{q}\Gamma^\mu Q|P\rangle = \epsilon_{\mu\nu\rho\sigma}\varepsilon^\nu p_P^\rho p_\gamma^\sigma F_V + i\big(\varepsilon^\mu p_P\cdot p_\gamma - p_\gamma^\mu \varepsilon\cdot p_P\big)F_A \tag{78}$$

The contribution in Fig. 1c depends on not only E_γ but also on p_ν and p_l. For simplicity, we treat this term separately, the form factors is written as

$$\begin{aligned}
&\langle\gamma|\bar{q}\Gamma^\mu Q|P\rangle + F_c p_P^\mu = \epsilon_{\mu\nu\rho\sigma}\varepsilon^\nu p_P^\rho p_\gamma^\sigma F_V + i\big(\varepsilon^\mu p_P\cdot p_\gamma - p_\gamma^\mu \varepsilon\cdot p_P\big)F_A + F_c p_P^\mu \\
&F_V = \frac{1}{(2\pi)^3}\frac{3}{\sqrt{6}}\int d^3k\Psi(k)\frac{1}{2\sqrt{p_{q0}p_{Q0}(p_{q0}+m_q)(p_{Q0}+m_Q)}}\frac{1}{m_P E_\gamma} \\
&\quad\times\big(2e_q C_1 m_Q - C_1^0 p_{q0}e_q + e_Q 2p_{q0}C_3\big) \\
&F_A = \frac{1}{(2\pi)^3}\frac{3}{\sqrt{6}}\int d^3k\Psi(k)\frac{1}{2\sqrt{p_{q0}p_{Q0}(p_{q0}+m_q)(p_{Q0}+m_Q)}}\frac{1}{m_P E_\gamma} \\
&\quad\times\left(2e_q C_1 m_Q - C_1^0 p_{q0}e_q - e_q\frac{2p_{q0}m_Q}{E_\gamma}C_2 - e_Q 2p_{q0}C_3 + e_Q\frac{2p_{q0}m_Q}{E_\gamma}C_4\right)
\end{aligned} \tag{79}$$

The relation $F_A = F_V$ at leading order is explicitly broken at order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ as expected [24]. We evaluate this integral using [16]

$$\begin{aligned}
&m_D = 1.9\ \mathrm{GeV}, \qquad m_B = 5.1\ \mathrm{GeV}, \qquad m_u = m_d = 0.08\ \mathrm{GeV} \\
&m_b = 4.98\ \mathrm{GeV}, \qquad m_c = 1.54\ \mathrm{GeV} \\
&\Lambda_{\mathrm{QCD}} = 200\ \mathrm{MeV}, \qquad \lambda_B = 2.8\ \mathrm{GeV}^{-1}, \qquad \lambda_D = 3.4\ \mathrm{GeV}^{-1}
\end{aligned} \tag{80}$$

The result of $F_{A,V}$ of $B\to\gamma e\nu_e$ is shown in Fig. 6, the $O(\Lambda_{\mathrm{QCD}}/E_\gamma)$ contribution is more important at the region $E_\gamma\to 0$ which is clearly shown in the figure. The numerical results of $F_{A,V}$ are inconvenient to use when calculate the decay widths. For simplicity, we use some simple forms to fit the numerical results. For the form factors at the order $O(\alpha_s(\Lambda_{\mathrm{QCD}}/m_Q)^0)$, we use the single-pole form

$$F_A^{\alpha_s}(E_\gamma) = F_V^{\alpha_s}(E_\gamma) = \frac{f(0)}{q^2 - m^{*2}} = \frac{f(0)}{m_P^2 - m^{*2} - 2m_P E_\gamma} \tag{81}$$

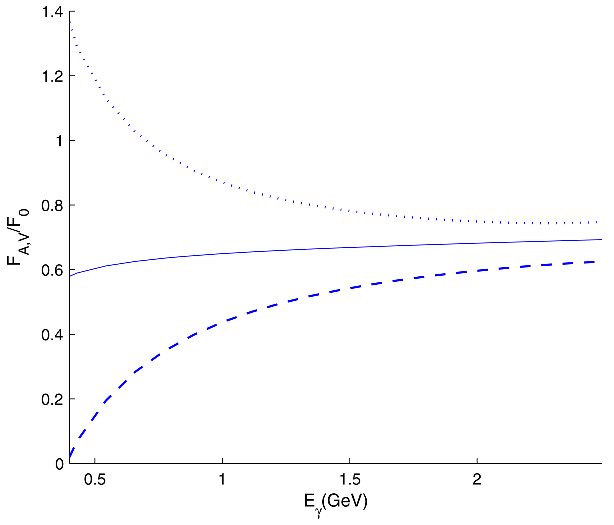

Fig. 6. Form factors of $B \to \gamma e \nu_e$ as functions of E_γ. The results are presented as the ratios of the form factors at 1-loop order to the one at the tree level and the leading order of the $\Lambda_{\mathrm{QCD}}/m_Q$ expansion, which is denoted as F_0. The solid line is $F_A/F_0 = F_V/F_0$, where $F_{A,V}$ are at the order of $O(\alpha_s(\Lambda_{\mathrm{QCD}}/m_Q)^0)$. The dotted line is F_V/F_0 and the dashed line is F_A/F_0, both at the order of $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$.

Table 1
The results of the parameters in the form factors in Eqs. (81) and (82).

	m^* (GeV)	$f(0)$ (GeV)	A_V	B_V	A_A	B_A
$B \to \gamma e \nu_e$	5.37	−0.63	0.27	0.45	0.32	−0.67
$D \to \gamma e \nu_e$	1.98	−0.15	−0.00095	−0.54	−0.27	−0.059

where $q = p_P - p_\gamma$. While up to the order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$, inspired by the form factors in Ref. [17] the form factors are fitted as

$$F_{A,V}^{\frac{\alpha_s}{m_Q}}(E_\gamma) = \left(A_{A,V}\frac{\Lambda_{\mathrm{QCD}}}{E_\gamma} + B_{A,V}\left(\frac{\Lambda_{\mathrm{QCD}}}{E_\gamma}\right)^2\right) \tag{82}$$

The predicted results for the form factors are more reliable at the region $E_\gamma \gg \Lambda_{\mathrm{QCD}}$ because we have neglected the higher order terms of $\Lambda_{\mathrm{QCD}}/E_\gamma$ in the numerical calculation, so we choose the region $E_\gamma > 2\Lambda_{\mathrm{QCD}}$ to fit the parameters in Eqs. (81) and (82). The fitting is given in Table 1 and shown in Figs. 7 and 8. On the other hand, F_c can be related to the decay constant [16,17] by

$$F_c p_P^\mu = ie\frac{p_l \cdot \varepsilon P_L^\mu}{p_l \cdot p_\gamma}\langle 0|\bar{q}\gamma_\mu(1-\gamma_5)Q|P\rangle = -ief_P p_P^\mu \frac{p_l \cdot \varepsilon}{p_l \cdot p_\gamma} \tag{83}$$

Using the fitted result of F_A, F_V, the result of F_c, and using the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements [25,26]

$$V_{\mathrm{cd}} = 0.226, \qquad V_{\mathrm{ub}} = 0.0047 \tag{84}$$

we obtain the result for the branching ratios. There are IR divergences in the radiative leptonic decays in the case that the photon is soft or the photon is collinear with the emitted lepton. The-

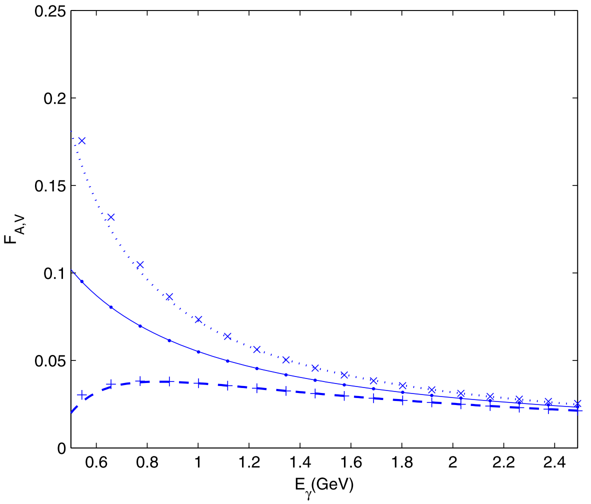

Fig. 7. Fit of the form factors of $B \to \gamma e \nu_e$. The solid line is for the result at the leading order of Λ_{QCD}/m_Q. The '×' points and the dotted line are for F_V while the dashed line and the '+' points are for F_A.

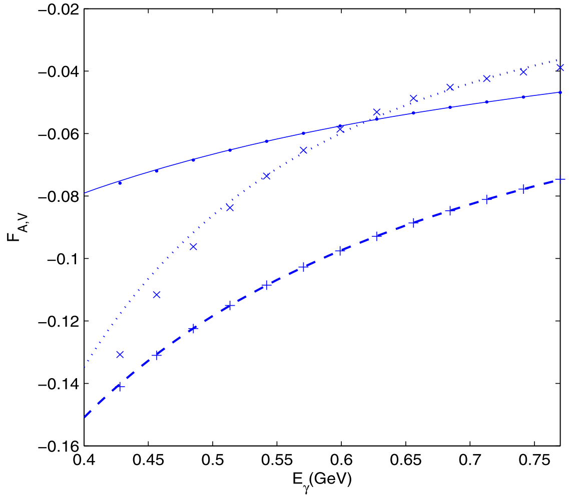

Fig. 8. Fit of the form factors of $D \to \gamma e \nu_e$. The solid line is for the result at the leading order of Λ_{QCD}/m_Q. The '×' points and the dotted line are for F_V while the dashed line and the '+' points are for F_A.

oretically this IR divergences can be canceled by adding the decay rate of the radiative leptonic decay with the pure leptonic decay rate, in which one-loop correction is included [27]. The radiative leptonic decay cannot be distinguished from the pure leptonic decay in experiment when the photon energy is smaller than the experimental resolution to the photon energy. So the decay rate of the radiative leptonic decay depend on the experimental resolution to the photon energy E_γ which is denoted by ΔE_γ. The dependence of the branching ratios on the resolution are listed in Table 2.

Using $\Delta E_\gamma = 10$ MeV [28], the branching ratios are given in Table 3. We find that, in general, the 1-loop results are smaller than the tree level results. The 1-loop correction is found to be

Table 2
The branching ratios with different photon resolution ΔE_γ.

ΔE_γ	$\mathrm{BR}(B \to e\nu_e\gamma)$	$\mathrm{BR}(D \to e\nu_e\gamma)$	ΔE_γ	$\mathrm{BR}(B \to e\nu_e\gamma)$	$\mathrm{BR}(D \to e\nu_e\gamma)$
5 MeV	1.77×10^{-6}	3.10×10^{-5}	20 MeV	1.56×10^{-6}	2.53×10^{-5}
10 MeV	1.66×10^{-6}	2.81×10^{-5}	25 MeV	1.53×10^{-6}	2.45×10^{-5}
15 MeV	1.60×10^{-6}	2.64×10^{-5}	30 MeV	1.48×10^{-6}	2.38×10^{-5}

Table 3
The branching ratios of the decay modes.

	$\mathrm{BR}_{O((\frac{\alpha_s \Lambda_{\mathrm{QCD}}}{m_Q})^0)}$	$\mathrm{BR}_{O(\alpha_s(\frac{\Lambda_{\mathrm{QCD}}}{m_Q})^0)}$	$\mathrm{BR}_{O(\frac{\alpha_s \Lambda_{\mathrm{QCD}}}{m_Q})}$
$B \to e\nu_e\gamma$	2.38×10^{-6}	1.01×10^{-6}	1.66×10^{-6}
$D \to e\nu_e\gamma$	9.62×10^{-6}	2.71×10^{-6}	2.81×10^{-5}

important. For B meson, the correction to the decay amplitude at the order $O(\Lambda_{\mathrm{QCD}}/m_Q)^0$ is about 10% to 30% due to the large logarithms.

The contribution of the order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$ are generally not negligible. For B meson, the correction of the order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contribution to the decay amplitude can be as large as 30%. For D mesons, the mass of c quark is not large enough, the order $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contributions is much more important, it is necessary to include higher order corrections in $\Lambda_{\mathrm{QCD}}/m_Q$ expansion.

8. Conclusion

In this paper, we study the factorization of the radiative leptonic decays of B^- and D^- mesons. Compared with the work in Ref. [8], the factorization is extended to include the $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contributions, and the transverse momentum is also considered. The factorization is proved explicitly at 1-loop order, the valid region of the factorization is extended. The hard kernel at order $O(\alpha_s \Lambda_{\mathrm{QCD}}/m_Q)$ is obtained. We use the wave function obtained in Ref. [16] to derive the numerical results. The branching ratios of $B^- \to \gamma e\bar{\nu}$ is found to be at the order of 10^{-6}, which is close to the previous works [7,17,29–31]. In the previous works, the results of D mesons are different from each other from 10^{-3} to 10^{-6}, our results agree with 10^{-5}.

We also find that the $O(\Lambda_{\mathrm{QCD}}/m_Q)$ contribution is very important even for B meson, the correction to the decay amplitude is about 20%–30%, which can affect the branching ratios about 50%. This is because of the importance of $O(\Lambda_{\mathrm{QCD}}/E_\gamma)$ contributions. In previous works, $O(\Lambda_{\mathrm{QCD}}/E_\gamma)$ contributions is neglected. For a typical region, $E_\gamma \sim m_Q/4$, which is also the leading region of the phase space of the tree level, the neglected contributions can be up to 20%. As a result, the correction to the branching ratios can be up to 40% at tree level.

Acknowledgements

This work is supported in part by the National Natural Science Foundation of China under contracts Nos. 11375088, 10975077, 10735080, 11125525.

Appendix A. IBP reductant relation of wave function

The integral is

$$\Phi_q^{(1)}(k_{\bar{q}},k_Q)=\int d^4x\int d^4y e^{ik_{\bar{q}}\cdot x}e^{ik_Q\cdot y}\langle 0|\bar{q}_{\bar{q}}(x)ig_s\int_y^x dz z_\mu A^\mu(z)Q(y)$$
$$\times ig_s\int d^4x_2\bar{q}_{\bar{q}}(x_2)\not{A}(x_2)q_{\bar{q}}(x_2)\big|\bar{q}^S(p_{\bar{q}}),Q^s(p_Q)\big\rangle \tag{A.1}$$

After variable substitution using $z=x+\alpha y$, we find

$$\int_y^x dz z_\mu A^\mu(z)=\int_0^1 d\big(y+\alpha(x-y)\big)_\mu A^\mu\big(y+\alpha(x-y)\big)$$
$$=\int_0^1 d\alpha(x-y)_\mu A^\mu\big(y+\alpha(x-y)\big) \tag{A.2}$$

After the contraction and then integral over x_2, p, the result is

$$\Phi_q^{(1)}\otimes T^{(0)}$$
$$=-C_F g_s^2\int d^4x\int d^4y\int_0^1 d\alpha\int\frac{d^dl}{(2\pi)^d}$$
$$\times e^{i y\dot{(}k_Q+\alpha l-p_Q)}e^{i x\dot{(}k_{\bar{q}}-\alpha l-p_{\bar{q}})}\bar{v}_{\bar{q}}\gamma\cdot(x-y)\frac{1}{l^2}\frac{(-\not{l}-\not{p}_{\bar{q}}+m_{\bar{q}})}{(l+p_{\bar{q}})^2-m_{\bar{q}}^2}T^{(0)}u_Q \tag{A.3}$$

The integral over x and $k_{\bar{q}}$ can be worked out term by term. However, we find IBP reduction relation [18] an elegant way to do so. Consider this integral

$$\int\frac{d^4k_q}{(2\pi)^4}\Gamma(k_q)\int d^4x e^{-i(p_{\bar{q}}+\alpha l-k_{\bar{q}})\cdot x} \tag{A.4}$$

It is unchanged when k_q is shifted, so under the infinitesimal transformation

$$k_q\to k_q+\beta K \tag{A.5}$$

The integral transforms as

$$\Gamma(k_q)\int d^4x e^{-i(p_{\bar{q}}+\alpha l-k_{\bar{q}})\cdot x}$$
$$\to\left(\beta K\cdot\frac{\partial}{\partial k_q}\right)\left(\Gamma(k_q)\int d^4x e^{-i(p_{\bar{q}}+\alpha l-k_{\bar{q}})\cdot x}\right) \tag{A.6}$$

The Lie algebra leads to

$$\int\frac{d^4k_q}{(2\pi)^4}\left(K\cdot\frac{\partial}{\partial k_q}\right)\left(\Gamma(k_q)\int d^4x e^{-i(p_{\bar{q}}+\alpha l-k_{\bar{q}})\cdot x}\right)=0 \tag{A.7}$$

so that

$$\int \frac{d^4k_q}{(2\pi)^4}\Gamma(k_q)K\cdot x\int d^4x e^{-i(p_{\bar{q}}+\alpha l-k_{\bar{q}})\cdot x}=iK\cdot\frac{\partial\Gamma(k_q)}{\partial k_q}\bigg|_{k_q=p_{\bar{q}}+\alpha l} \tag{A.8}$$

So after integrate over x, $k_{\bar{q}}$, y and k_Q, the result is

$$\begin{aligned}
&\Phi_q^{(1)}\otimes T^{(0)}\\
&=ig_s^2C_F\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2}\bar{v}_{\bar{q}}\gamma^\rho\frac{(\not{l}+\not{p}_{\bar{q}}-m_{\bar{q}})}{(l+p_{\bar{q}})^2-m_{\bar{q}}^2}\int_0^1 d\alpha\left(\frac{\partial T^{(0)}}{\partial k_q^\rho}-\frac{\partial T^{(0)}}{\partial k_Q^\rho}\right)\bigg|_{k_q=k',k_Q=K'}u_Q\\
&k'=p_{\bar{q}}+\alpha l,\quad K'=p_Q-\alpha l
\end{aligned} \tag{A.9}$$

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